MTH 3440, Partial Differential Equations—Final Exam **May 11, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 12:50 pm.

1. Which of the following are solutions of the partial differential equation $u_t = \alpha^2 u_{xx}$? (Note: No boundary conditions; no initial condition.) *Give your reasoning.*

   (a) $u(x,t) = e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x)$, where λ is an arbitrary constant.

   (b) $u(x,t) = 3x - 2$.

   (c) $u(x,t) = 2e^{\lambda^2\alpha^2 t}\tan 2\lambda x$, where λ is an arbitrary constant.

   (d) $u(x,t) = 4\alpha^2 e^{4\alpha^2 t}\cosh 2x$.

   Solution:

   (a) This is a solution because

   $$u_t - \alpha^2 u_{xx} = -\lambda^2\alpha^2 e^{-\lambda^2\alpha^2 t}(\cos\lambda x - \sin\lambda x) - \alpha^2(-\lambda^2\cos\lambda x + \lambda^2\sin\lambda x) = 0.$$

   (b) This is a solution because both u_t and u_{xx} vanish everywhere—making $u_t = \alpha^2 u_{xx}$.

   (c) This is not a solution. $u_t = 2\alpha^2\lambda^2\tan 2\lambda x$, but $\alpha^2 u_{xx} = 16\alpha^2\lambda^2 e^{\lambda^2\alpha^2 t}\tan 2\lambda x\sec^2 2\lambda x$.

   (d) This is a solution because

   $$u_t - \alpha^2 u_{xx} = 16\alpha^4 e^{4\alpha^2 t}\cosh 2x - \alpha^2[4\alpha^2 e^{4\alpha^2 t}(4\cosh 2x)] = 0.$$

2. Solve:

   $$\begin{aligned} u_{tt} &= 4u_{xx}, & -\infty < x < \infty, 0 < t \\ u(x,0) &= 2\sin x\cos x, & -\infty < x < \infty \\ u_t(x,0) &= \cos x, & -\infty < x < \infty. \end{aligned}$$

   Solution: Solutions to the wave equation $u_{tt} = cu_{xx}$ on $-\infty < x < \infty$ have the form

   $$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi,$$

   where f and g are the initial data for u and u_t respectively. The solution to the given problem is therefore

   $$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin 2(x-2t) + \sin 2(x+2t)] + \frac{1}{4}\int_{x-2t}^{x+2t}\cos\xi\,d\xi \\ &= \frac{1}{2}[\sin 2(x-2t) + \sin 2(x+2t)] + \frac{1}{4}\sin\xi\Big|_{x-2t}^{x+2t} \\ &= \frac{1}{2}[\sin 2(x-2t) + \sin 2(x+2t)] + \frac{1}{4}[\sin(x+2t) - \sin(x-2t)]. \end{aligned}$$

3. Solve the Cauchy problem:

   $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - u = 0; \tag{1}$$

   $$\tag{2}$$

   $$u(x,1) = f(x), \tag{3}$$

where f is a continuously differentiable function.

Solution: The characteristic equation of the PDE is

$$y' = \frac{y}{x},$$

whose solution can be written in the form $y/x = c$. Thus, we choose new coordinates

$$\xi = x$$
$$\eta = \frac{y}{x},$$

in which to write the PDE. Putting $w(\xi, \eta) = u(x, y)$, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial w}{\partial \xi} - \frac{y}{x^2}\frac{\partial w}{\partial \eta} = \frac{\partial w}{\partial \xi} - \frac{\eta}{\xi}\frac{\partial w}{\partial \eta}$$
$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{x}\frac{\partial w}{\partial \eta} = \frac{1}{\xi}\frac{\partial w}{\partial \eta}$$

Substituting these relations into (1) yields

$$\xi\left(\frac{\partial w}{\partial \xi} - \frac{\eta}{\xi}\frac{\partial w}{\partial \eta}\right) + \xi\eta\left(\frac{1}{\xi}\frac{\partial w}{\partial \eta}\right) - w = 0,$$

which reduces to

$$\xi\frac{\partial w}{\partial \xi} - w = 0,$$

or

$$\frac{\partial}{\partial \xi}\left(\frac{w}{\xi}\right) = 0.$$

Thus, $w = \xi g(\eta)$, where g is an arbitrary (well-behaved) function depending upon the parameter η only. It now follows that our solution $u(x, y)$ must have the form

$$u(x, y) = xg\left(\frac{y}{x}\right).$$

But according to the Cauchy data, $u(x, 1) = f(x)$, and from this we conclude that $f(x) = u(x, 1) = xg(1/x)$. Thus, $g(1/x) = f(x)/x$, or $g(z) = zf(1/z)$. Finally, we see that

$$u(x, y) = yf\left(\frac{x}{y}\right).$$

4. Show how to transform the IBVP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,\ 0 < t < \infty, \tag{4}$$
$$u(0, t) = \cos \pi t, \qquad 0 < t < \infty, \tag{5}$$
$$u(1, t) = \sin \pi t, \qquad 0 < t < \infty, \tag{6}$$
$$u(x, 0) = x, \qquad 0 < x < 1, \tag{7}$$

into an IBVP with homogeneous boundary conditions. (*Do not* attempt to solve the transformed problem.)

Solution: Let us put

$$w(x,t) = (1-x)\cos \pi t + x \sin \pi t,$$

and

$$v(x,t) = u(x,t) - w(x,t).$$

Then $u(x,t) = v(x,t) + w(x,t)$, so

$$u_t = v_t + w_t = v_t + \pi(x-1)\sin \pi t + \pi x \cos \pi t$$

and

$$u_{xx} = v_{xx} + w_{xx} = v_{xx}.$$

Consequently, the partial differential equation (4) becomes

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \pi(1-x)\sin \pi t - \pi x \cos \pi t.$$

From (5) and our definition of v, we find that

$$v(0,t) = 0.$$

Similarly, (6) becomes

$$v(1,t) = 0.$$

Finally, from (7) we find that

$$v(x,0) = x - (1-x) = 2x - 1.$$

Thus, the given IBVP transforms into the IBVP with homogeneous boundary conditions:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \pi(1-x)\sin \pi t - \pi x \cos \pi t, \qquad 0 < x < 1,\ 0 < t < \infty, \tag{8}$$

$$v(0,t) = 0, \qquad 0 < t < \infty, \tag{9}$$

$$v(1,t) = 0, \qquad 0 < t < \infty, \tag{10}$$

$$v(x,0) = 2x - 1, \qquad 0 < x < 1. \tag{11}$$

5. Show how to use separation of variables to find the Fourier expansion of the solution for the interior Dirichlet problem on the polar unit disk $\{(r,\theta)\colon 0 \le r < 1\}$:

$$\nabla^2 u = 0, \qquad 0 \le r < 1$$
$$u(1,\theta) = g(\theta),$$

where g is continuous and periodic of period 2π. You may assume that the general solution to the Euler equation $r^2 R'' + rR' - \lambda^2 R = 0$ is $R = a + b\ln r$ when $\lambda = 0$ and $R = ar^{\lambda} + br^{-\lambda}$ when $\lambda \neq 0$. It may also be helpful to recall that, in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Solution: An outline of the solution is on page 263 of the text, and a more detailed discussion should be in your classroom notes.