Farlow's Problem 18.3

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We are to solve the IBVP[1]:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty; 0 < t < \infty \tag{1}$$

$$u_x(0,t) = 0 \qquad 0 < t < \infty \tag{2}$$

$$u(x,0) = f(x) \qquad 0 \le x < \infty \tag{3}$$

$$u_t(x,0) = g(x) \qquad 0 \le x < \infty \tag{4}$$

Solution: We write the d'Alembert solution,

$$u(x,t) = \varphi(x+ct) + \psi(x-ct), \tag{5}$$

and solve for the unknown functions φ, ψ.

Taking partial derivatives with respect to t in (5) leads us to

$$u_t(x,t) = c\varphi'(x+ct) - c\psi'(x-ct), \text{ whence} \tag{6}$$

$$u_t(x,0) = c\varphi'(x) - c\psi'(x). \tag{7}$$

But, according to (4), this means that when $v \ge 0$ we must take

$$c\varphi'(v) - c\psi'(v) = g(v), \tag{8}$$

so that when $z \ge 0$ we will have

$$\int_0^z [\varphi'(v) - \psi'(v)]\,dv = \frac{1}{c}\int_0^z g(v)\,dv \tag{9}$$

$$[\varphi(z) - \psi(z)] - [\varphi(0) - \psi(0)] = \frac{1}{c}\int_0^z g(v)\,dv \tag{10}$$

$$\varphi(z) - \psi(z) = \frac{1}{c}\int_0^z g(v)\,dv + [\varphi(0) - \psi(0)]. \tag{11}$$

[1]Farlow's Problem 18.3 gives $g(x) \equiv 0$; we solve the more general problem here.

On the other hand, combining (3) with (5) gives

$$\varphi(z) + \psi(z) = f(z), \tag{12}$$

also when $z \geq 0$.

Adding (11) to (12) and dividing the result by 2, we learn that

$$\varphi(z) = \frac{1}{2} f(z) + \frac{1}{2c} \int_0^z g(v)\, dv + \frac{1}{2}[\varphi(0) - \psi(0)] \tag{13}$$

when $z \geq 0$. On the other hand, if we subtract (11) from (12) and divide by 2, we learn that

$$\begin{aligned} \psi(z) &= \frac{1}{2} f(z) - \frac{1}{2c} \int_0^z g(v)\, dv - \frac{1}{2}[\varphi(0) - \psi(0)] && (14) \\ &= \frac{1}{2} f(z) - \frac{1}{2c} \int_0^z g(v)\, dv - \frac{1}{2}[\varphi(0) - \psi(0)] && (15) \end{aligned}$$

when $z \geq 0$.

When $x - ct \geq 0$, we combine (5), (13), and (15) to write

$$\begin{aligned} u(x,t) &= \varphi(x+ct) + \psi(x-ct) && (16) \\ &= \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(v)\, dv + \frac{1}{2}[\varphi(0) - \psi(0)] \\ &\qquad + \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(v)\, dv - \frac{1}{2}[\varphi(0) - \psi(0)] && (17) \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(v)\, dv. && (18) \end{aligned}$$

We make use of the boundary condition (2) to see what we should do for $x - ct < 0$. Taking partial derivatives with respect to x in (5), we obtain

$$u_x(x,t) = \varphi'(x+ct) + \psi'(x-ct), \tag{19}$$

Substituting (2) into (19), we find that

$$\begin{aligned} 0 &= u_x(0,t) && (20) \\ &= \varphi'(ct) + \psi'(-ct), && (21) \end{aligned}$$

when $ct > 0$, and from this we conclude that when $\xi > 0$ we must take $\psi'(-\xi) = -\varphi'(\xi)$. Thus, when $z > 0$ we must have

$$\begin{aligned} \int_0^z \psi'(-\xi)\, d\xi &= -\int_0^z \varphi'(\xi)\, d\xi && (22) \\ -\psi(-z) + \psi(0) &= -\varphi(z) + \varphi(0), && (23) \\ \psi(-z) &= \varphi(z) + \psi(0) - \varphi(0). && (24) \end{aligned}$$

Equivalently, when $z < 0$ we must have

$$\psi(z) = \varphi(-z) + \psi(0) - \varphi(0). \tag{25}$$

We combine (25) and (13), now, to learn that when $z < 0$ we must put

$$\begin{aligned}
\psi(z) &= \varphi(-z) + \psi(0) - \varphi(0) && (26)\\
&= \frac{1}{2}f(-z) + \frac{1}{2c}\int_0^{-z} g(v)\,dv + \frac{1}{2}[\varphi(0) - \psi(0)] + \psi(0) - \varphi(0) && (27)\\
&= \frac{1}{2}f(-z) + \frac{1}{2c}\int_0^{-z} g(v)\,dv - \frac{1}{2}[\varphi(0) - \psi(0)]. && (28)
\end{aligned}$$

Consequently, when $x - ct < 0$, we must combine (5), (13), and (28). For such values of x and t, we get

$$\begin{aligned}
u(x,t) &= \varphi(x+ct) + \psi(x-ct) && (29)\\
&= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(v)\,dv + \frac{1}{2}[\varphi(0) - \psi(0)]\\
&\qquad + \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(v)\,dv - \frac{1}{2}[\varphi(0) - \psi(0)] && (30)\\
&= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(v)\,dv + \frac{1}{2}f(|x-ct|) - \frac{1}{2c}\int_{ct-x}^{0} g(v)\,dv\\
&= \frac{1}{2}[f(x+ct) + f(|x-ct|)] + \frac{1}{2c}\int_0^{x+ct} g(v)\,dv + \frac{1}{2c}\int_{x-ct}^{0} g(-v)\,dv\\
&= \frac{1}{2}[f(x+ct) + f(|x-ct|)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(|v|)\,dv. && (31)
\end{aligned}$$

It now follows that the solution to the given IBVP is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(|x-ct|)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(|v|)\,dv. \tag{32}$$