**MTH 3240, Differential Equations—Problem Set 1** **Due Feb. 9, 2009**

**Instructions:** Work the following problems and submit your solutions at the beginning of class on Monday, Feb. 9, 2009. You may use any resources you like, but the solutions you submit must be solutions that you, yourself, have written from the understanding that you have gained from the resources you use. If I have doubts about your understanding of a solution you have presented, I may ask you to come to my office and elaborate it orally.

1. Work Problem 5 of Lesson 1.

   **Solution:** If $u_{xy}(x,y) = 0$, then integrating partially with respect to $y$ yields $u_x(x,y) = \Phi(x)$ for some unknown function $\Phi$ of $x$—which we must treat as a constant for a partial integration with respect to $y$. Integrating $\Phi(x)$ partially with respect to $x$ now gives $u(x,y) = \varphi(x) + \psi(y)$, where $\psi(y)$ is an unknown function of $y$ and $\varphi(x)$, the integral of the unknown function $\Phi(x)$, is an unknown function of $x$.

2. Work Problem 3 of Lesson 2.

   **Solution:** In the steady-state, we have $u_t(x,t) = 0$, so the partial differential equation $u_t = \alpha^2 u_{xx} - \beta u$ becomes $0 = \alpha^2 u_{xx} - \beta u$. We can treat this latter equation as a linear, homogeneous, constant coefficients, ordinary differential equation because there is no $t$-dependence. The solution must therefore have the form $u(x) = c_1 e^{(\sqrt{\beta}/\alpha)x} + c_2 e^{(-\sqrt{\beta}/\alpha)x}$. Applying the boundary conditions, we find that

$$c_1 = \frac{1}{e^{\sqrt{\beta}/\alpha} + 1},$$
$$c_2 = \frac{e^{\sqrt{\beta}/\alpha}}{e^{\sqrt{\beta}/\alpha} + 1}.$$

   Thus, the steady-state solution is

$$\begin{aligned}
u(x) &= \frac{e^{(\sqrt{\beta}/\alpha)x} + e^{(\sqrt{\beta}/\alpha)(1-x)}}{e^{\sqrt{\beta}/\alpha} + 1} \qquad (1)\\
&= \frac{e^{-(\sqrt{\beta}/\alpha)}(e^{(\sqrt{\beta}/\alpha)} - 1)(e^{(\sqrt{\beta}/\alpha)x} + e^{(\sqrt{\beta}/\alpha)(1-x)})}{e^{-(\sqrt{\beta}/\alpha)}(e^{(\sqrt{\beta}/\alpha)} - 1)(e^{\sqrt{\beta}/\alpha} + 1)}\\
&= \frac{e^{(\sqrt{\beta}/\alpha)x} - e^{-(\sqrt{\beta}/\alpha)x} + e^{(\sqrt{\beta}/\alpha)(1-x)} - e^{-(\sqrt{\beta}/\alpha)(1-x)}}{e^{\sqrt{\beta}/\alpha} - e^{-\sqrt{\beta}/\alpha}}\\
&= \frac{\frac{1}{2}[e^{(\sqrt{\beta}/\alpha)x} - e^{-(\sqrt{\beta}/\alpha)x}] + \frac{1}{2}[e^{(\sqrt{\beta}/\alpha)(1-x)} - e^{-(\sqrt{\beta}/\alpha)(1-x)}]}{\frac{1}{2}[e^{\sqrt{\beta}/\alpha} - e^{-\sqrt{\beta}/\alpha}]}\\
&= \frac{\sinh[(\sqrt{\beta}/\alpha)x] + \sinh[(\sqrt{\beta}/\alpha)(1-x)]}{\sinh[(\sqrt{\beta}/\alpha)]} \qquad (2)
\end{aligned}$$

   So the solution in the form (1) (which is certainly adequate) really does reduce to the form (2), which is what appears in the answers in the back of the book.

3. Work Problem 3 of Lesson 3.

   **Solution:** See the solution at the back of the book.

4. Work Problem 6 of Lesson 5.

   **Solution:** We must find the coefficients $a_k$ in the Fourier sine expansion

$$x - x^2 = \sum_{k=1}^{\infty} a_k \sin k\pi x.$$

This means that we must evaluate the integral $2\int_0^1 (x - x^2)\sin k\pi x\,dx$, for each positive integer $k$. Integrating repeatedly by parts (three repetitions), we find that

$$\begin{aligned} 2\int_0^1 (x - x^2)\sin k\pi x\,dx & \\ &= 2\left[-\frac{(x - x^2)}{k\pi}\cos k\pi x + \frac{(1 - 2x)}{k^2\pi^2}\sin k\pi x - \frac{2}{k^3\pi^3}\cos k\pi x\right]\Bigg|_0^1 \\ &= -\frac{4}{k^3\pi^3}(\cos k\pi - \cos 0) \\ &= \frac{4[1 - (-1)^k]}{k^3\pi^3}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \frac{4[1 - (-1)^k]}{k^3\pi^3} e^{-k^2\pi^2 t}\sin k\pi x \\ &= \frac{8}{\pi^3}e^{-\pi^2 t}\sin \pi x + \frac{8}{27\pi^3}e^{-9\pi^2 t}\sin 3\pi x \\ &\qquad + \frac{8}{125\pi^3}e^{-25\pi^2 t}\sin 5\pi x + \frac{8}{343\pi^3}e^{-49\pi^2 t}\sin 7\pi x + \cdots . \end{aligned}$$

Can you make an argument using symmetry showing why the even-numbered terms *should* drop out?