MTH 3240, Partial Differential Equations—Problem Set 2 **Due Feb. 25, 2009**

Instructions: Work the following problems and submit your solutions at the beginning of class on Monday, Feb. 25, 2009. You may use any resources you like, but the solutions you submit must be solutions that you, yourself, have written from the understanding that you have gained from the resources you use. If I have doubts about your understanding of a solution you have presented, I may ask you to come to my office and elaborate it orally.

1. Let f be a function which is defined and indefinitely differentiable on the interval $[0, \infty)$. The *Laplace transform*, $\mathcal{L}[f]$, of f is the function given by

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \tag{1}$$

provided that the improper integral converges. Show how to derive the formulae:

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0), \tag{2}$$

$$\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0), \text{ and} \tag{3}$$

$$\mathcal{L}[f^{(3)}](s) = s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0). \tag{4}$$

You may assume that all of the Laplace transforms you encounter converge and that

$$\lim_{t\to\infty} e^{-st}f^{(k)}(t) = 0,$$

for all $k \geq 0$, at least when $s > 0$.

Solution: Integrating by parts, we have

$$\begin{aligned}\mathcal{L}[f'](s) &= \int_0^\infty e^{-st}f'(t)\,dt \\ &= e^{-st}f(t)\Big|_0^\infty + \int_0^\infty se^{-st}f(t)\,dt \\ &= -f(0) + s\int_0^\infty e^{-st}f(t)\,dt = s\mathcal{L}[f](s) - f(0).\end{aligned}$$

Then, applying (2) twice, we obtain

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) \\ &= s[s\mathcal{L}[f](s) - f(0)] - f'(0) = s^2\mathcal{L}[f](s) - sf(0) - f'(0).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{L}[f'''](s) &= s\mathcal{L}[f''](s) - f''(0) \\ &= s[s^2\mathcal{L}[f](s) - sf(0) - f'(0)] - f''(0) \\ &= s^3\mathcal{L}[f](s) - s^2f(0) - sf'(0) - f''(0).\bullet\end{aligned}$$

2. Use the method of Lesson 6, Farlow, to show how to transform the IBVP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,\ 0 < t < \infty, \tag{5}$$

$$u(0,t) = \cos \pi t, \qquad 0 < t < \infty, \tag{6}$$

$$u(1,t) = \sin \pi t, \qquad 0 < t < \infty, \tag{7}$$

$$u(x,0) = x, \qquad 0 < x < 1, \tag{8}$$

into an IBVP with homogeneous boundary conditions.

Solution: Let us put

$$w(x,t) = (1-x)\cos \pi t + x \sin \pi t,$$

and

$$v(x,t) = u(x,t) - w(x,t).$$

Then $u(x,t) = v(x,t) + w(x,t)$, so

$$u_t = v_t + w_t = v_t + \pi(x-1)\sin \pi t + \pi x \cos \pi t$$

and

$$u_{xx} = v_{xx} + w_{xx} = v_{xx}.$$

Consequently, the partial differential equation (5) becomes

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \pi(1-x)\sin \pi t - \pi x \cos \pi t.$$

From (6) and our definition of v, we find that

$$v(0,t) = 0.$$

Similarly, (7) becomes

$$v(1,t) = 0.$$

Finally, from (8) we find that

$$v(x,0) = x - (1-x) = 2x - 1.$$

Thus, the given IBVP transforms into the IBVP with homogeneous boundary conditions:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \pi(1-x)\sin \pi t - \pi x \cos \pi t, \qquad 0 < x < 1,\ 0 < t < \infty, \tag{9}$$

$$v(0,t) = 0, \qquad 0 < t < \infty, \tag{10}$$

$$v(1,t) = 0, \qquad 0 < t < \infty, \tag{11}$$

$$v(x,0) = 2x - 1, \qquad 0 < x < 1. \tag{12}$$

3. Find the solution for the IBVP of Problem 2, above.

Solution: Let us write $V_k(t)$ for the k-th term of the finite Fourier sine transform of the solution $v(x,t)$ taken in the variable x. Then, as we have seen in class, $\mathcal{F}_s[v_t](k) = V_k'(t)$ and

$$\begin{aligned}\mathcal{F}_s[v_{xx}](k) &= 2k\pi[v(0,t) - (-1)^k v(1,t)] - k^2\pi^2 V_k(t) \\ &= -k^2\pi^2 V_k(t),\end{aligned}$$

where we have obtained the latter equation from the former by applying (10) and (11). The right-hand side of (9) transforms as

$$\begin{aligned}\mathcal{F}_s[\pi(1-x)\sin\pi t - \pi x\cos\pi t](k) &= 2\int_0^1 [\pi(1-x)\sin\pi t - \pi x\cos\pi t]\,dx \\ &= \frac{2}{k}[\sin\pi t + (-1)^k\cos\pi t].\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{F}_s[v(x,0)](k) &= \mathcal{F}_s[2x-1](k) \\ &= 2\int_0^1 (2x-1)\sin k\pi x\,dx \\ &= \frac{2}{k\pi}[(-1)^{k+1} - 1].\end{aligned}$$

Consequently, the IBVP problem (9–12) transforms to the sequence of single-variable initial value problems

$$V_k'(t) + k^2\pi^2 V_k(t) = \frac{2}{k}[\sin\pi t + (-1)^k\cos\pi t], \tag{13}$$

$$V_k(0) = \frac{2}{k\pi}[(-1)^{k+1} - 1]. \tag{14}$$

Let us write

$$\alpha_k = k^2\pi^2, \tag{15}$$

$$\beta_k = \frac{2}{k}, \tag{16}$$

$$\gamma_k = (-1)^k\frac{2}{k}, \tag{17}$$

and

$$v_k = \frac{2}{k\pi}[(-1)^{k+1} - 1], \tag{18}$$

so that these IVPs can be written more concisely as

$$\begin{aligned}V_k' + \alpha_k V_k &= \beta_k\sin\pi t + \gamma_k\cos\pi t, \\ V_k(0) &= v_k.\end{aligned}$$

Supplying an integrating factor $e^{\alpha_k t}$, we find that for each k we must have

$$
\begin{aligned}
\frac{d}{dt}\left[e^{\alpha_k t} V_k(t)\right] &= \beta_k e^{\alpha_k t}\sin\pi t + \gamma_k e^{\alpha_k t}\cos\pi t;\\
\int_0^t \frac{d}{dt}\left[e^{\alpha_k \tau} V_k(\tau)\right]\, d\tau &= \int_0^t \left[\beta_k e^{\alpha_k \tau}\sin\pi\tau + \gamma_k e^{\alpha_k \tau}\cos\pi\tau\right] d\tau;\\
e^{\alpha_k t} V_k(t) - v_k &= \frac{\pi\beta_k - \alpha_k\gamma_k + e^{\alpha_k t}[(\alpha_k\gamma_k - \pi\beta_k)\cos\pi t + (\alpha_k\beta_k + \pi\gamma_k)\sin\pi t]}{\pi^2 + \alpha_k^2}.
\end{aligned}
$$

Thus, the solution to the kth initial value problem is given by

$$
V_k(t) = \left(v_k + \frac{\pi\beta_k - \alpha_k\gamma_k}{\pi^2 + \alpha_k^2}\right)e^{-\alpha_k t} + \frac{(\alpha_k\gamma_k - \pi\beta_k)\cos\pi t + (\alpha_k\beta_k + \pi\gamma_k)\sin\pi t}{\pi^2 + \alpha_k^2}. \tag{19}
$$

We have now obtained the finite Fourier sine transform of the solution $v(x,t)$ of the transformed IBVP, and we need only apply the inverse finite Fourier sine transform and add $w(x,t) = (1 - x)\cos\pi t + x\sin\pi t$ to the result in order to obtain $u(x,t)$. We conclude that

$$
u(x,t) = (1-x)\cos\pi t + x\sin\pi t + \sum_{k=1}^{\infty} V_k(t)\sin k\pi x, \tag{20}
$$

where $V_k(t)$ is given by (19) and the quantities α_k, β_k, γ_k and v_k are given by (15–18).