MTH 3240, Partial Differential Equations—Problem Set 3 **Due April 8, 2009**

Instructions: Work the following problems and submit your solutions at the beginning of class on Wednesday, April 8, 2009. You may use any resources you like, but the solutions you submit must be solutions that you, yourself, have written from the understanding that you have gained from the resources you use. If I have doubts about your understanding of a solution you have presented, I may ask you to come to my office and elaborate it orally.

1. Work Problem 1 of Lesson 18.

   Solution: We are to solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; && 0 \le x \le \infty, 0 < t < \infty \\ u(0,t) &= 0; && 0 < t < \infty \\ u(x,0) &= xe^{-x^2}; && 0 < x < \infty \\ u_t(x,0) &= 0; && 0 < x < \infty. \end{aligned}$$

Putting $f(x) = xe^{-x^2}$, $g(x) = 0$, we substitute into the D'Alembert solution (see equation (17.8), page 133, of Farlow) to find that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi \\ &= \frac{1}{2}\left[(x-ct)e^{-(x-ct)^2} + (x+ct)e^{-(x+ct)^2}\right], \end{aligned}$$

as long as $x \ge ct$.

When $x < ct$, we observe that $x - ct < 0$, and we use the relation $u(x,t) = \varphi(x-ct) + \psi(x+ct)$ to write

$$0 = u(0,t) = \varphi(-ct) + \psi(+ct),$$

from which it follows that for $w > 0$ we must take

$$\varphi(-w) = -\psi(w) = -we^{-w^2}.$$

This means that for $z < 0$ we must put

$$\varphi(z) = -\psi(-z) = ze^{-z^2}.$$

Thus, when $x - ct < 0$, we put

$$u(x,t) = \frac{1}{2}\left[(x-ct)e^{-(x-ct)^2} + (x+ct)e^{-(x+ct)^2}\right].$$

Our solution is therefore given by

$$u(x,t) = \frac{1}{2}\left[(x-ct)e^{-(x-ct)^2} + (x+ct)e^{-(x+ct)^2}\right].$$

2. Solve the semi-infinite string problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; & 0 \le x \le \infty, 0 < t < \infty \\ u(0,t) &= \sin t; & 0 < t < \infty \\ u(x,0) &= 0; & 0 < x < \infty \\ u_t(x,0) &= 0; & 0 < x < \infty. \end{aligned}$$

Solution: Putting $f(x) = 0$, $g(x) = 0$, we substitute into the D'Alembert solution (see equation (17.8), page 133, of Farlow) to find that

$$u(x,t) = 0$$

as long as $x \ge ct$.

When $x < ct$, we observe that $x - ct < 0$, and we use the relation $u(x,t) = \varphi(x-ct) + \psi(x+ct)$ to write

$$\sin t = u(0,t) = \varphi(-ct) + \psi(+ct),$$

or

$$\varphi(-ct) = \sin t - \psi(ct) = \sin t.$$

From this it follows that for $w > 0$ we must take

$$\varphi(-w) = -\psi(w) + \sin(w/c) = \sin(w/c).$$

This means that for $z < 0$ we must put

$$\varphi(z) = \sin(-z/c) = -\sin(z/c).$$

Thus, when $x - ct < 0$, we put

$$u(x,t) = -\sin[(x-ct)/c].$$

Our solution is therefore given by

$$u(x,t) = \begin{cases} 0, & \text{when } x - ct \ge 0, \\ -\sin[(x-ct)/c], & \text{when } x - ct < 0. \end{cases}$$

3. Solve the IBVP associated with the *hammered string:*

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; & -L \le x \le L, 0 < t < \infty \\ u(x,0) &= 0; & 0 < t < \infty \\ u_t(x,0) &= \begin{cases} 0, & \text{when } \frac{L}{10} < |x| \\ 1, & \text{when } |x| \le \frac{L}{10}; \end{cases} & \\ u(-L,t) &= 0; & 0 < t < \infty \\ u(L,t) &= 0; & 0 < t < \infty. \end{aligned}$$

Solution: The standard separation of variables procedure yields solutions to the partial differential equation of the form

$$u(x,t) = (A\cos\lambda ct + B\sin\lambda ct)(C\cos\lambda x + D\sin\lambda x).$$

The requirement that $u(x,0) = 0$ for all x then implies that $A = 0$. From the conditions $u(-L,t) = 0$ and $u(L,t) = 0$, both for all positive t, we then find that

$$(C\cos\lambda L - D\sin\lambda L)\sin\lambda ct = 0$$

and

$$(C\cos\lambda L + D\sin\lambda L)\sin\lambda ct = 0.$$

Adding these equations together, we find that we must have $C\cos\lambda L = 0$, so that either $C = 0$ or $\lambda = (2k+1)\pi/(2L)$ for some integer k. However, if we subtract the first from the second we find that $\sin\lambda L = 0$, and this implies that either $D = 0$ or $\lambda = k\pi/L$ for some integer k. We thus obtain two families of solutions:

$$u_k(x,t) = \sin\frac{(2k+1)\pi ct}{2L}\cos\frac{(2k+1)\pi x}{2L},$$

and

$$v_k(x,t) = \sin\frac{k\pi ct}{L}\sin\frac{k\pi x}{L},$$

with $k = 0, 1, \ldots$ in each case.

We have yet to make use of the second initial condition. We have

$$\begin{aligned}\frac{\partial u_k}{\partial t} &= \frac{(2k+1)\pi c}{2L}\cos\frac{(2k+1)\pi ct}{2L}\cos\frac{(2k+1)\pi x}{2L},\\ \frac{\partial v_k}{\partial t} &= \frac{k\pi c}{L}\cos\frac{k\pi ct}{L}\sin\frac{k\pi x}{L}.\end{aligned}$$

Now the functions $\partial v_k/\partial t$ are all odd functions, and our initial velocity is an odd function. We therefore discard all of the functions v_k and look for a solution which is of the form

$$u(x,t) = \sum_{k=0}^{\infty} c_k u_k(x,t).$$

Of course, we must have

$$u_t(x,0) = \sum_{k=0}^{\infty} c_k \frac{(2k+1)\pi c}{2L}\cos\frac{(2k+1)\pi x}{2L}.$$

Consequently, we must have, for each $k = 0, 1, \ldots$,

$$\begin{aligned}\int_{-L}^{L} u_t(x,0)\cos\frac{(2k+1)\pi x}{2L}\,dx &= \sum_{j=0}^{\infty} c_j \frac{(2j+1)\pi c}{2L}\int_{-L}^{L}\cos\frac{(2j+1)\pi x}{2L}\cos\frac{(2k+1)\pi x}{2L}\,dx.\\ &= c_k\frac{(2k+1)\pi c}{2},\end{aligned}$$

where we have used the orthogonality relations

$$\int_{-L}^{L} \cos\frac{(2j+1)\pi x}{2L} \cos\frac{(2k+1)\pi x}{2L}\, dx = \begin{cases} 0 & \text{if } k \neq j \\ L & \text{if } k = j. \end{cases}$$

We thus see that

$$c_k = \frac{2}{(2k+1)\pi c} \int_{-L}^{L} u_t(x,0) \cos\frac{(2k+1)\pi x}{2L}\, dx.$$

However, $u_t(x,0) = \chi_{[-L/10,L/10]}(x)$, so

$$\begin{aligned} \int_{-L}^{L} u_t(x,0) \cos\frac{(2k+1)\pi x}{2L}\, dx &= \int_{-L/10}^{L/10} \cos\frac{(2k+1)\pi x}{2L}\, dx \\ &= \frac{4L\sin[(2k+1)\pi/20]}{(2k+1)\pi}, \end{aligned}$$

and thus, finally,

$$c_k = \frac{8L\sin[(2k+1)\pi/20]}{[(2k+1)\pi]^2 c}.$$

The solution we seek is

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8L\sin[(2k+1)\pi/20]}{[(2k+1)\pi]^2 c} \sin\frac{(2k+1)\pi ct}{2L} \cos\frac{(2k+1)\pi x}{2L}.$$